COMPUTER-ALGEBRA METHODS FOR INVESTIGATION OF HEREDITARY OPERATORS OF HIGHER ORDER SOLITON EQUATIONS

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The hereditariness of recursion operators is discussed for some 5th order nonlinear partial differential equations as well as for several coupled systems. Consequences of the hereditary property are surveyed. An outline of the corresponding computer algebra proofs (based on the formula manipulation systems MAPLE and MACSYMA) is given. Several new hierarchies of completely integrable systems are presented.

1. Introduction

A linear operator \( \Phi : L \rightarrow L \) on a Lie algebra \( L \) is said to be hereditary \([5,7]\) if

\[
\Phi^2[H_1, H_2] + [\Phi H_1, \Phi H_2] = \Phi[H_1, \Phi H_2] + \Phi[\Phi H_1, H_2]
\]

for all \( H_1, H_2 \) in the Lie algebra. If the Lie structure under consideration is the algebra of \( \mathbb{C}^\infty \)-vectorfields on some manifold then hereditary operators play a major role in describing the structural properties of completely integrable nonlinear evolution equations \([6,18]\) (see section 3 for a brief survey). In ref. \([8]\) it was claimed that the recursion operators coming out of the bi-Hamiltonian structure of the Caudrey–Dodd–Gibbon–Sawada–Kotera equation \([1,19]\) and the Kupershmidt equation \([4]\) are hereditary. At that time it was impossible (for reasons to be explained later) to present a direct calculation for this claim, so structural arguments were used to “prove” this claim. These arguments were mainly based on the assumption that spectral gradient operators (i.e. squared eigenfunction operators for isospectral flows of the Schrödinger operator) of the eigenvalue problems given by Lax pairs are automatically hereditary. Actually this assumption is tacitly used quite often in the literature and explicitly it is presented in ref. \([3]\). However, the “proof” of this statement is generally based on the erroneous assumption that the eigenvectors of the spectral gradient operator are in some sense dense. Unfortunately, at least in case of completely integrable Hamiltonian systems, this never happens. This is due to the fact that the generators of the canonical one-parameter symmetry groups are orthogonal to the gradients of the conservation laws given by the discrete points of the spectrum of the Lax operator. So, even in the transparent finite dimensional case where the manifold under consideration is the \( N \)-soliton manifold, there is a nonempty orthogonal complement of the spectral gradient operator on the manifold under consideration. An example where the spectral gradient operator of some eigenvalue problem was believed \([9,17]\) to be a non-hereditary recursion operator for the corresponding isospectral flows is given by the Hirota–Satsuma equation \([13]\). In this case the recursion operator coming out of the bi-Hamiltonian formulation \([9]\) of this system is a spectral gradient operator \([17]\) of the Lax pair formulation \([2]\) of the system. Both authors (ref. \([9]\) as well as ref. \([17]\)) independently believed to have shown by a simple calculation that this operator...
cannot be hereditary. Later it turned out that in both cases these calculations were incorrect (a fact which was due to the computational difficulties). So the problem whether the recursion operators of these systems (as well as of the above mentioned 5th order equations) are hereditary or not is still open.

In this note we first show why this is a difficult computational problem. We discuss what is gained by knowing the answer to this question. Then we briefly describe how the definite answer concerning the hereditariness of these operators has been achieved. At the end we give several new hereditary operators which were found by computer algebra methods. These operators yield new hierarchies of completely integrable flows on infinite dimensional manifolds.

2. The computational problem

The recursion operators of the CDGSK \[1,19\]

\[
\frac{d}{dt} u = H_{CD}(u) = u_{xxxx} + \frac{1}{2} uu_{xxx} + \frac{1}{2} u_x u_{xx} + \frac{3}{2} u^2 u_x
\]  

(2)

and the Kupershmidt equation \[4\]

\[
\frac{d}{dt} u = H_K(u) = u_{xxxx} + \frac{1}{2} uu_{xxx} u + \frac{3}{2} u_x u_x + \frac{1}{2} u^2 u_x
\]  

(3)

are given by \[8\]

\[
\Phi_{CD}(u) = \Theta_{CD}(u) J_{CD}(u)
\]  

(4.1)

and

\[
\Phi_K(u) = \Theta_K(u) J_K(u),
\]  

(4.2)

where

\[
\Theta_{CD}(u) = \Theta_K(u) = D^3 + u D + Du,
\]  

(4.3)

\[
J_{CD}(u) = 2D^3 + D^2 u D^{-1} + D^{-1} u D^2 + \frac{1}{2} (u^2 D^{-1} + D^{-1} u^2),
\]  

(4.4)

\[
J_K(u) = 2D^3 + \frac{1}{2} (u D + Du) + D^2 u D^{-1} + D^{-1} u D^2 + u^2 D^{-1} + D^{-1} u^2,
\]  

(4.5)

and where \(D = d/dx\) and \(D^{-1} = \int_{-\infty}^x \cdot d\xi\). Due to the bi-Hamiltonian structure \[9\] of these equations the corresponding hierarchies of vectorfields commuting with the flows are given by \(\Phi^n(u) u_x\) and \(\Phi^n(u) H_{CD}(u)\) (or \(\Phi^n(u) H_K(u)\), respectively), \(n \in \mathbb{N}\) (see ref. \[8\]). We know \[5\] that an operator \(\Phi\) is hereditary if and only if for all vectorfields \(v, w\) the expression

\[
S(v, w) = \Phi'(u)[\Phi(u)v] w - \Phi(u) \Phi'(u)[v] w
\]  

(5)

is symmetric in \(v\) and \(w\). Here, as usual, \(\Phi'(u)[w]\) denotes the variational derivative

\[
\Phi'(u)[w] = \frac{\partial}{\partial \epsilon}
\]  

\(\bigg|_{\epsilon=0}\Phi(u + \epsilon w).

Looking at the explicit structure of the operator \(\Phi_{CD}\) one discovers that \(\Phi_{CD}(u)[w]\) is a differential operator having 22 terms. Since \(\Phi_{CD}\) itself has 15 terms the operator \(\Phi'(u)[\Phi(u)w]\) has now 330 terms. Hence \(S(v, w)\) has 660 terms, and the symmetry condition to be checked amounts to the verification whether a differential expression of about 1300 terms equals zero. For the Kupershmidt equation we have the same task for an expression of about 2600 terms. And these are not the most complicated operators being candidates for the hereditary property.
Even if one splits these expressions into different powers of $u$ it is completely hopeless to check the hereditariness by hand, especially since equating these expressions to zero involves a delicate application of the product rule and the rule for integration by parts (see for example the simple case of the KdV and other equations, where this has been done explicitly in [5]). Of course, there are cases of known higher order hereditary operators [7], but usually for these cases the hereditariness is not proved by explicit calculation but rather by application of structural results applied to factors. Alas, this is not possible in our case and therefore the enormousness of the necessary computation prevents a simple answer to the question whether or not these operators are hereditary.

Another problem which presents considerable computational difficulties is the proof of hereditariness of the recursion operator for the two-component Hirota–Satsuma system [13]

\begin{equation}
\frac{d}{dt} \psi = H_{HS}(u, \psi) = \begin{bmatrix}
\frac{1}{2}u_{xxx} + 3uu_x - 3\varphi \varphi_x \\
-\varphi_{xxx} - 3u\varphi_x
\end{bmatrix}.
\end{equation}

This recursion operator is given by

\begin{equation}
\Phi_{HS}(u, \psi) = \Theta_{HS}(u, \psi)J_{HS}(u, \psi),
\end{equation}

where

\begin{equation}
\Theta_{HS}(u, \psi) = \begin{bmatrix}
\frac{1}{2}D^3 + Du + uD & D\varphi + \varphi D \\
D\varphi + \varphi D & D^3 + 2Du + 2uD
\end{bmatrix},
\end{equation}

\begin{equation}
J_{HS}(u, \psi) = \begin{bmatrix}
D + 2D^{-1}u + 2uD^{-1} & -2\varphi D^{-1} \\
-2D^{-1}\varphi & -2D
\end{bmatrix}.
\end{equation}

Here also the number of terms to be considered is tremendous.

It seems most desirable to perform the necessary calculations for these problems by computer. At first glance the proper way to do this seems to be the following: consider it as a formal language problem with terminal element zero where the product rules of the grammar are given by the usual arithmetic operations, together with integration by parts and the product rule of differentiation. Unfortunately, this is not so simple since product rule and integration by parts give rise to a context-sensitive grammar. Notwithstanding these difficulties a computer program based on symbolic manipulation systems can perform the task. As an application of this program package we obtain that all the operators mentioned above, i.e. (4.1), (4.2) as well as (7.1), are hereditary. New operators found to be hereditary will be presented in section 5.

3. Advantages of hereditariness (a brief survey)

In order to put the problem of this paper in proper perspective we like to give a brief survey on the advantages of hereditary structure:

A. If the hereditary property of some operator $\Phi$ is granted then commutativity of the vectorfields in the generated hierarchy is usually a complete triviality (see ref. [7]). This is so because eq. (1) implies that we have

\[ [\Phi^n H_1, \Phi^m H_2] = 0 \]

whenever

\[ [H_1, H_2] = 0, \]
and
\[ \Phi[H, H_1] = [H, \Phi H_1], \]  
\[ \Phi[H, H_2] = [H, \Phi H_2] \]  
for all \( H \) in the Lie-algebra. The properties (8.1) and (8.2) are just the same as saying that \( \Phi \) is a recursion operator for the flows given by the generators \( H_1 \) and \( H_2 \). Hence, \( \Phi \) generates out of suitable starting points commutative Lie algebras. For those cases satisfying all the assumptions of this paragraph (see ref. [8]), this amounts to the construction of an infinite dimensional symmetry group of the system. Or, in other words, we have found hierarchies of commuting flows
\[ u_n = K_n(u), \text{ where } K_n = \Phi^n H_1, \]  
\[ u_n = G_n(u), \text{ where } G_n = \Phi^n H_2 \]  
by application of \( \Phi \). Since all these flows do admit infinite dimensional symmetry groups (given by the other flows of the hierarchy) they are usually said to be completely integrable.

The best known example is the hereditary operator of the Korteweg–de Vries equation
\[ \Phi(u) = D^2 + 2u + 2DuD^{-1}, \]  
where (8.1) and (8.2) are fulfilled for the vectorfield given by the generator \( u_x = H_1(u) = H_2(u) \), of the translation group. Recursive application of \( \Phi \) then gives access to most of the relevant quantities (conservation laws, symmetry groups, soliton solutions etc.).

B. If the hereditary operator comes out of a bi-Hamiltonian formulation [9] of the system (as it is the case for the systems under consideration [8]) then, via Noether’s theorem, all the symmetry generators given by the recursion correspond to gradients of conservation laws. Furthermore, these conservation laws are in involution. This well-known fact has been discovered by several authors for, in the bihamiltonian case, our notion of hereditariness coincides with that of compatibility (given by Gelfand–Dorfmann [12]) as well as with the notion of Nijenhuis structure (given by Magri [16]).

All the consequences of hereditariness we presented so far are concerned (more or less) with the action variables of the system. But there are other consequences. Let us mention first two of a more theoretical nature.

C. If the hereditary property of a recursion operator resulting from a bi-Hamiltonian formulation of a dynamical system is verified then immediately infinitely many different Hamiltonian formulations (hence infinitely many different Poisson brackets [6]) are established.

D. If one has hereditary structure with free parameters then one can use them to construct new hereditary operators in the way described in ref. [7]. Since \( \Phi_{\text{CD}} \) as well as \( \Phi_K \) can be transformed into operators involving two parameters, this recipe can be used for these cases. But the dependence on these parameters is not a linear one, therefore things are too involved to elaborate them any further in this short note.

E. Apart from these more theoretical properties the hereditariness gives in addition information about time dependent conserved quantities and therefore information about the angle variables of the system. A consequence of this is that our operators are yielding a complete description of the action-angle variables (at least in case of restrictions onto invariant finite dimensional submanifolds (soliton manifolds)). In order to explain this we need some notation (see ref. [10]).

A vectorfield \( G(u, t) \), which explicitly depends on the parameter \( t \) (time), is said to be a time-dependent symmetry generator of the flow
\[ \frac{d}{dt} u = H(u) \]
if

$$G_t = [H, G].$$

(9.2)

Here $G_t$ denotes the partial derivative and $[ , ]$ is the commutator on the Lie algebra of vector fields. If the hereditary operator $\Phi(u)$ (not depending explicitly on $t$) is a recursion operator for (9.1), i.e. if we have $\Phi[H, A] = [H, \Phi A]$ for all vector fields $A$, then (1) obviously implies that $\Phi$ maps time-dependent symmetry generators onto time-dependent symmetry generators.

Now, assume that a time-dependent symmetry $G_0(u, t) = \Gamma_0(u) + tR_0(u)$ is given which is of polynomial degree 1 in $t$. Then

$$G_n(u, t) = \Phi^n(u)G_0(u, t) = \Gamma_n(u) + tR_n(u)$$

(10)

constitutes a hierarchy of time-dependent symmetries of the same polynomial degree in $t$. Via the Hamiltonian formulation of the system one can transform these quantities into conserved covariants and furthermore into conservation laws

$$P_n(u, t) = \Pi_n(u) + tF_n(u), \quad n \in \mathbb{N},$$

if these conserved covariants do have potentials. And obviously the quantities

$$\alpha_n(u) = \Pi_n(u)/F_n(u)$$

are angle variables since $t - \alpha_n(u)$ remains constant under the flow of the system.

So, for finding the angle variables of the system we need some time-dependent symmetry generator as starting point. For the eqs. (2), (3) and (6), as well as for the new systems given in section 5, these are easily found since the equations have similar scaling properties as the KdV. To be precise, consider the vectorfield

$$\Gamma_0(u) = 2u + xu_x,$$

(11)

then

$$[H_{CD}, \Gamma_0] = -5H_{CD},$$

(11.1)

$$[H_K, \Gamma_0] = -5H_K,$$

(11.2)

$$[H_{HS}, \Gamma_0] = -3H_{HS}.$$  

(11.3)

Hence

$$G_0 = \Gamma_0 + tH$$

(12)

provides respectively for $H = 5H_{CD}$, $H = 5H_K$ and $H = 3H_{HS}$ time-dependent symmetry-generators for the CDGSK, Kupershmidt equation and the Hirota–Satsuma system. Now, we can use the corresponding recursion operators to generate out of the starting points hierarchies of such symmetry generators. Unfortunately, these do not correspond to conservation laws since the appropriate conserved covariants do not admit potentials. But restriction to those finite dimensional manifolds which support the soliton solutions overcomes this obstacle. Hence, we find the action-angle representation for these manifolds explicitly. This method has been successfully applied to find complete diagonalizations for anisotropic nonlinear spin-chains with cyclic boundary conditions. For this and other systems the explicit computations for this representation will be reported elsewhere.
4. The computation

Based on computer algebra we have written computer programs in order to check whether or not (5) gives a symmetric function in \( v \) and \( w \). It is a straightforward task to define procedures which calculate expression (5) in terms of arbitrary functions \( v(x), w(x), u(x) \) (which may split up into components \( v.1(x), v.2(x), \ldots \)), a formal differential operator \( D \) and a formal integration operator \( D^{-1} \). This is not at all the crucial point in the calculation, the above procedures can be given in a couple of lines. But the result \( S(v, w) - S(w, v) \) is a huge sum of symbols generated by \( D, D^{-1} \) and multiplications with the functions \( u(x), v(x), w(x) \) and their derivatives, which (hopefully) vanishes. It is not trivial to check whether this expression really equals zero! This is due to the appearance of the integration operator making the above expression a context sensitive problem. E.g. the sum

\[
D^{-1}vD^{-1}uw_{xx} + D^{-1}u,vw + D^{-1}uw_{x}w + D^{-1}vD^{-1}u_{x}w_{x} - uww
\]

(13)
does vanish, but it takes a subtle integration by parts to see this. But this is a trivial example compared to expressions of about 1000 to 4000 terms coming up in the problems under consideration. Hence one needs a “normalized” form of each of the summands in (13), which then can be compared and reduced by the standard simplifiers of the algebra packages. The formal differential operator does not give any problem as to this, just using the product rule will definitely handle all summands generated only by \( D \) und multiplications with arbitrary functions. All the algebra packages provide a fast differentiation using the product rule, hence one need not define a formal derivation \( D \) (which would be very simple). The crucial procedure is \( D^{-1} \), which has to know about integration by parts. The built-in integration algorithms of the algebra systems, although extremely valuable for other problems, are definitely not suitable for this application, hence new procedures had to be written. But due to the context sensitivity it is not easy to define what a “normalized” form of an integration should be. Fortunately expression (5) has a useful special feature: there are two linear entries \( v \) and \( w \).

Based on this fact we used a formal integration routine having the following algebraic properties:

i) \( D^{-1} \) is a linear operator.

ii) \( DD^{-1} = D^{-1}D = \text{identity} \). Note that this is satisfied for the operator \( \int_{-\infty}^{x} \cdot d\xi \) provided that all the entries vanish at \( x = -\infty \).

iii) The argument of \( D^{-1} \) is checked for the occurrence of the function \( v(x) \) and \( w(x) \), respectively. Then the numbers of derivatives attached to \( v \) are counted. Using integration by parts the expression is converted (if possible) into summands which contain only \( v(x) \) but no derivatives or further integrations of \( v \), e.g.

\[
D^{-1}uw_{xx} \rightarrow w_{x} - u_{x}v + D^{-1}u_{xx}v,
D^{-1}uD^{-1}v \rightarrow (D^{-1}u)(D^{-1}v) - D^{-1}vD^{-1}u.
\]

(14)
The strategy of “normalizing” integrations indicated by iii) is first used with \( v \) then with \( w \), in every go a considerable number of summands cancel such that the computer can identify \( S(v, w) - S(w, v) \) as zero at last. Now, the computation was only a matter of routine if one were able to handle in a precise way a huge amount of data consisting of integro-differential expressions in the formal variables \( u, v, w \).

We implemented three different programs handling these data:

i) in PASCAL [14],

ii) in MAPLE, the formula manipulation system developed by the MAPLE-group of the University of Waterloo [11],

iii) in MACSYMA [15].

The running time for our Pascal programs was enormous, a fact which may have come out of our inexperience in programming such problems. More seriously, the answer was unconclusive since we had no
efficient way of normalizing nonlinear expressions in our formal variables. All this was done only by approximation. The result was — within our limits of accuracy — that $\Phi_{CD}$ must be hereditary. The operator $\Phi_K$ we did not try because there was no hope that this could be done in reasonable CPU-time.

With MACSYMA as well as with MAPLE we obtained (in reasonable time) the conclusive answers that all the operators under consideration were indeed hereditary.

Compared to our PASCAL-program both program packages (MAPLE as well as MACSYMA) reduced the running time for $\Phi_{CD}$ by a factor of over 250 (a number which partly shows our inexperience when writing the PASCAL programs).

MAPLE provided the fastest programs. Furthermore, it allowed to run the problem on smaller sized workstations (SUN 3/50 and Cadmus 9230) instead of our VAX 11/750.

5. New hierarchies of completely integrable systems

With the computer algebra methods described above we have found several new hereditary operators for coupled systems with two or three components depending on the variable $x \in \mathbb{R}$. The operators are of a highly complex nature and there is virtually no hope that the hereditary property can be seen by inspection. All completely integrable systems which can be constructed by these operators are related to meaningful dynamical systems in Mathematical Physics. However, their physical significance will be discussed elsewhere.

A. Two component systems

The following operator is hereditary

$$\Phi_1(u, \varphi) = \begin{bmatrix} u + 2\varphi D^{-1}u & 1 + \varphi D^{-1} \\ D + 2u_x D^{-1}u - 4u \varphi D^{-1}u & -u + u_x D^{-1} - 2u \varphi D^{-1} \end{bmatrix}. $$

The following (even more complicated) operator also has been demonstrated to be hereditary

$$\Phi_2(u, \varphi) = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \tag{16}$$

where

$$\Psi_{11}(u, \varphi) = 2D^2 + \frac{1}{2}(u_x + \varphi)^2 - \cos(u) - \frac{1}{2}(u_x + \varphi)D^{-1}(u_{xx} + \varphi_x - \sin(u)), \tag{17.1}$$

$$\Psi_{12}(u, \varphi) = 2D + \frac{1}{2}(u_x + \varphi)D^{-1}(u_x + \varphi), \tag{17.2}$$

$$\Psi_{21} = \Psi_{12}(u, \varphi)(D^2 - \cos u) + \Psi_{11}(u, \varphi)[K_0], \tag{17.3}$$

$$\Psi_{22} = \Psi_{11}(u, \varphi) + \Psi_{12}(u, \varphi)[K_0], \tag{17.4}$$

and where expressions like $\Psi'(u, \varphi)[K_0]$ stand for the two-component variational derivative

$$\Psi'(u, \varphi)[K_0] = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \{ \Psi(u + \epsilon \varphi, \varphi + \epsilon(u_{xx} - \sin(u))) \} \tag{18.1}$$

into the direction of the vectorfield

$$K_0(u, \varphi) = \begin{bmatrix} \varphi \\ u_{xx} - \sin u \end{bmatrix}. \tag{18.2}$$

For both of these operators eqs. (8.1) and (8.2) hold with

$$H_1 = H_2 = \begin{bmatrix} u_x \\ \varphi_x \end{bmatrix}. \tag{19}$$
Hence, the nonlinear partial differential equations given by the following two-component systems

\[
\begin{bmatrix}
  u \\
  \varphi
\end{bmatrix}
= \Phi(u, \varphi)^n \begin{bmatrix}
  u_x \\
  \varphi_x
\end{bmatrix}, \quad n \in \mathbb{N}
\]  

are found to be completely integrable.

B. Three component systems

We consider completely integrable dynamical systems of the form

\[
\frac{d}{dt} \begin{bmatrix}
  u \\
  \varphi \\
  \psi
\end{bmatrix}
= K(u, \varphi, \psi)
\]  

given by hereditary operators. For example, we found that the following operators are hereditary

\[
\Phi_3(u, \varphi, \psi)
\]

\[
= \begin{bmatrix}
  4u^2 + 4\varphi D^{-1}(\psi + 6u^4) & -4\varphi D^{-1}\varphi & 1 + 4\varphi D^{-1}u \\
  D + 4\varphi D^{-1}(\psi + 6u^4) & -2u^2 - 4\varphi D^{-1}\varphi & 4\varphi D^{-1}u \\
  4u\varphi + 4(u_x - 6u^2\varphi)D^{-1}(\psi + 6u^4) & D - 4\varphi - 4(u_x - 6u^2\varphi)D^{-1}\varphi & -2u^2 + 4(u_x - 6u^2\varphi)D^{-1}u
\end{bmatrix}
\]

and

\[
\Phi_4(u, \varphi, \psi)
\]

\[
= \begin{bmatrix}
  4u + 12\varphi D^{-1}u & 0 & 1 + 2\varphi D^{-1} \\
  D + 12\varphi D^{-1}u & -2u & 2\varphi D^{-1} \\
  2\psi + 12u_x D^{-1}u - 72u\varphi D^{-1}u & D - 2\varphi & -2u + 2u_x D^{-1} - 12u\varphi D^{-1}
\end{bmatrix}
\]

are hereditary.

As before any system with the right-hand side of (21) given by

\[
K(u, \varphi, \psi) = \Phi(u, \varphi, \psi)^n \begin{bmatrix}
  u_x \\
  \varphi_x \\
  \psi_x
\end{bmatrix}, \quad n \in \mathbb{N}
\]

is completely integrable.

The amount of data to be handled for the proof of hereditaryness of these operators is roughly of the same size as before. We needed

- words = 00088572, time = 0019 s for the operator given in (15)
- words = 05788707, time = 1273 s for the operator given in (16)
- words = 00402587, time = 0085 s for the operator given in (22)
- words = 00121557, time = 0025 s for the operator given in (23)

on our SUN 3/50 workstation (one word = 4 Byte). Due to the good garbage collection of MAPLE 3.3 the necessary disk space for the data never exceeded 2 MBytes. The times are system-time.

6. Comparison with other work

Pioneering work in the determination of symmetries and conservation laws for dynamical systems by computer algebra methods has been done by Schwarz [20–22]. Let us point out what the differences between his work and ours is. The notion of symmetry is in a certain sense dual to the notion of conservation law (Noether's theorem). For flows on finite dimensional manifolds the conservation laws are
described by partial differential equations (namely those having the given flow as equation for the characteristic lines). By solving these partial differential equations, for a polynomial ansatz in the field variables, Schwarz determines the conservation laws and the symmetries by use of a powerful REDUCE-package. It turns out that the running times for finding these solutions are considerable (due to the complexity of the problem). So, on first view it seems hopeless to transfer these methods to infinite dimensional manifolds, where the corresponding description of conservation laws is a variational equation (something like a partial differential equation in infinitely many independent variables). But this first impression is misleading because some *special* symmetries are still described by partial differential equations. These are those which depend on the field variables in such a way that only $t$- and $x$-derivatives enter linearly (apart from an arbitrary dependence in the field variable itself). These are the symmetries going back to the fundamental work of Sophus Lie. They are called Lie-symmetries or rather Lie-point symmetries in contrast to arbitrary symmetries which are nowadays mostly called Lie–Bäcklund-symmetries. The programs developed by Schwarz determine the Lie-point symmetries for a class of infinite dimensional systems. Our approach determines, for certain systems, in addition all Lie–Bäcklund-symmetries. The difference between the two symmetry notions is considerable. For example, in the KdV-case the group of Lie-point symmetries is four dimensional, whereas that of Lie–Bäcklund symmetries is infinite dimensional. Our program works without solving any partial differential equation. So it looks as if this approach were more powerful. But this is not correct. Schwarz’s method works in all cases even when the system under consideration is far from being completely integrable. Whereas in our case a powerful additional algebraic structure (hereditary property), coming from complete integrability, is plugged into the procedure. The advantage is that this additional structure reduces the running time considerably so that also infinite dimensional cases can be treated in all generality. The disadvantage is that it is only suitable for a very restricted, but nevertheless interesting, class of equations, namely the completely integrable ones.

References